UPPER BOUNDS FOR MODULI OF CONTOUR INTEGRALS

Ref: Complex Variables by James Ward Brown and Ruel V. Churchil

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41. UPPER BOUNDS FOR MODULI OF CONTOUR INTEGRALS

Theorem

Let C denote a contour $z = z(t)$ $(a \le t \le b)$. Let M be a non-negative constant such that $|f(z)| \le M$. Then $\left| \int_C f(z)\,dz \right| \le M\,L$, where L is the length of the contour.

Proof

We know that

$$\left| \int_C f(z)\,dz \right| = \left| \int_a^b [z(t)]\,z'(t)\,dt \right| \leq \int_a^b \left| f[z(t)] \right| \left| z'(t) \right| dt$$

$\therefore$ For any non-negative constant M such that the values of f on C satisfy the inequality |f(z)| ≤ M.

$$\therefore \left| \int_C f(z)\,dz \right| \leq M \int_a^b \left| z'(t) \right| dt$$

$\therefore$ The integral on the right here represents the length of L of the contour, it follows that the modulus of the value of the integral of f along C does not exceed ML.

$$\therefore \left| \int_C f(z)\,dz \right| \leq \text{ML}.$$

Example 1

Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant prove that

$$\left| \int_C \frac{z+4}{z^3-1} dz \right| \leq \frac{6\pi}{7}$$

Solution

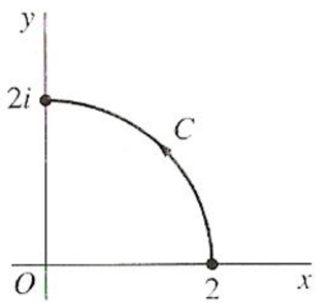

FIGURE 45

If z is a point on C, then $|z| = 2$.

Now $|z + 4| \leq |z| + |4| = 2 + 4 = 6$

Now $|z^3 - 1| \geq \left| |z^3| - |1| \right| = |8 - 1| = 7$

$$\Rightarrow \frac{1}{|z^3 - 1|} \leq \frac{1}{7}$$

$$\Rightarrow \frac{|z+4|}{|z^3-1|} \le \frac{6}{7}$$

We know that $\left|\int_C f(z)\,dz\right| \le ML$

$$\therefore \left|\int_C \frac{z+4}{z^3-1}\right| dz \le \left|\frac{z+4}{z^3-1}\right| L \le \frac{6}{7} L$$

$$\Rightarrow \left|\int_C \frac{z+4}{z^3-1}\right| dz \le \frac{6\pi}{7}$$

Example 2 Given that C_R is the semicircular path $z = Re^{i\theta}$ $(0 \le \theta \le \pi)$ and $z^{1/2}$ denotes the branch $z^{1/2} = \sqrt{r}\,e^{\frac{i\theta}{2}}$ $\left(r>0, -\frac{\pi}{2}<\theta<\frac{3\pi}{2}\right)$ of the square root function. Without actually finding the value of the integral, show that

$$\lim_{R\to\infty}\int_{C_R}\frac{z^{\left(\frac{1}{2}\right)}}{z^2+1}dz=0.$$

Solution

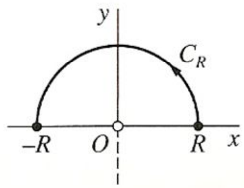

FIGURE 46

Given | z | = R > 1.

Now, $$\left| z^{\frac{1}{2}} \right| = \left| \sqrt{R}\, e^{\frac{i\theta}{2}} \right|$$

$$= \left| \sqrt{R} \right| \left| e^{\frac{i\theta}{2}} \right|$$

$$= \sqrt{R}\left(\left| \cos\frac{\theta}{2} + i\sin\frac{\theta}{2} \right| \right)$$

$$= \sqrt{R}\left(\sqrt{\cos^2\frac{\theta}{2} + i\sin^2\frac{\theta}{2}}\right) = \sqrt{R}$$

$$|z^2 + 1| \geq ||z^2| - 1| = R^2 - 1.$$

$$\Rightarrow \frac{1}{|z^2+1|} \leq \frac{1}{R^2-1}$$

$\therefore$ At points on C_R, $\left|\dfrac{z^{\frac{1}{2}}}{z^2+1}\right| \leq \dfrac{\sqrt{R}}{R^2-1} = M$

The length of C_R is $L = \pi R$.

$$\therefore \left| \int_C \frac{z^{\frac{1}{2}}}{z^2+1} dz \right| \leq ML$$

$$= \frac{\sqrt{R}}{R^2-1} \pi R$$

$$= \frac{\sqrt{R}}{R^2-1} \pi R \cdot \frac{1/R^2}{1/R^2} = \frac{\pi/\sqrt{R}}{\left(1 - \frac{1}{R^2}\right)}$$

$$\therefore \lim_{R \to \infty} \int_{C_R} \frac{z^{\left(\frac{1}{2}\right)}}{z^2+1} dz = 0$$